College Guild
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THE NUMBERS GAME

Unit 4 of 4

Beyond the four basic processes, there are other expressions one needs to understand to gain facility in math. These include exponents. Exponents are numbers written above and to the right of any ordinary number. They express the number of times a number is multiplied by itself.

For example, 5^2 is read "five to the second power" or "five squared". It means 5 x 5. In the expression "5 x 5" we are multiplying two numbers; each number is called a *factor*. In "5 x 5" then, there are two factors, each of which is a 5. In 5^2, the exponent "2" tells the number of factors of the base "5" that are to be multiplied together.

So, 5^3 means 5 x 5 x 5. This is also called "five cubed". The exponent (in this case 3) tells the number of 5's to multiply together. Here are some other examples:

$4^2 = 4 \times 4 = 16$ $6^3 = 6 \times 6 \times 6 = 216$ $2^4 = 2 \times 2 \times 2 \times 2 = 16$ $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

$8^5 = 8 \times 8 \times 8 \times 8 \times 8 = 4{,}096$ $7^3 = 7 \times 7 \times 7 = 343$

Notice how fast these numbers go up. This is what "increasing exponentially" means.

1. Find the following:

3^4 = **5^3 =** **9^3 =** **10^4 =**

2. Notice the number of zeroes in 10^4. With that in mind, what is 10^6?

The number of zeroes in any power of ten is the same as the exponent to which 10 is raised. This is a basic property of number systems. It may intrigue you to know how other number systems work, but it is not essential. Knowing basic squares and cubes is helpful, so your next task is to complete the tables below.

3. Table of Squares

1^2 =		**5^2 =**		**9^2 =**		**13^2 =**		**17^2 =**	
2^2 =		**6^2 =**		**10^2 =**		**14^2 =**		**18^2 =**	
3^2 =		**7^2 =**		**11^2 =**		**15^2 =**		**19^2 =**	
4^2 =		**8^2 =**		**12^2 =**		**16^2 =**		**20^2 =**	

4. Table of Cubes

1^3 =		2^3 =		3^3 =		4^3 =		5^3 =	
6^3 =		7^3 =		8^3 =		9^3 =		10^3 =	

Notice that one to any power is one. And of course, zero to any power is zero. Another set of numbers worth noticing is the powers of 2.

5. Find the following. (The first one, 2^0 has been done for you.)

$2^0 = 1$ **$2^3 =$** **$2^6 =$**

$2^1 =$ **$2^4 =$** **$2^7 =$**

$2^2 =$ **$2^5 =$** **$2^8 =$**

With this in mind, here is a famous problem:

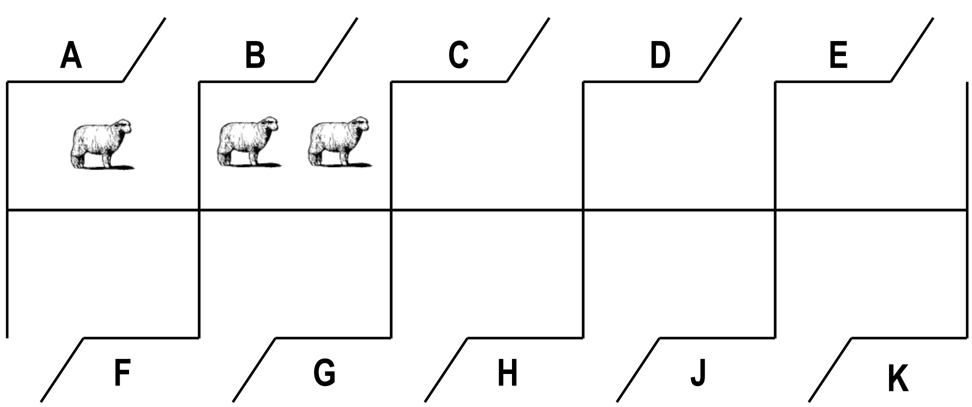
ONE THOUSAND SHEEP

A farmer has a thousand sheep. He wants to build pens and place the sheep in the pens so that anyone can come along and say, "I'd like 23 sheep," and the farmer can let out the correct number of sheep by opening up exactly the right gates. If, for example, a customer wanted just one sheep, there would have to be a pen with just one sheep in it. Of course, if a customer wanted two sheep, the farmer would need a pen with two sheep in it. Once the farmer opens a gate, all the sheep in that pen will come out; he can't empty part of a pen.

6. Would the farmer need a separate pen with 3 sheep in it? How about 4 sheep?

7. How would the farmer get 6 sheep?

8. Write the number of sheep (you do NOT have to draw them!) the farmer must put in each pen to enable him to open gates and get out any exact number of sheep from 1 up to 1,000.



9. What gates would the farmer open to get 13 sheep? How many sheep would come out of each pen that he opened?

10. To get 48 sheep? List the gates he needs to open, and how many sheep will come out of each pen. Now, add up the number of sheep that came out of each opened pen to check your work.

11. To get 73 sheep? List the gates opened, the number of sheep from each of the opened pens, and show the sum of your numbers.

12. Do the same for 86 sheep.

13. Do the same for 127 sheep.

14. Do the same for 259 sheep.

15. Do the same for 683 sheep.

16. How many sheep are in the last pen? (Be careful…the answer isn't especially obvious.)

If you have put the right number of sheep in each pen, you will have written the powers of 2 up to and including 2^8. But if you wrote the number for 2^9 in the last pen (K), you are not right.

17. Why would it be incorrect to write the number for 2^9 in the last pen (K)?

This problem of the thousand sheep is good for seeing some mathematical thinking. It is not of particular practical value (what farmer ever had exactly 1,000 sheep, and how can you get a sheep to do anything right in the first place?). But math is full of fun questions to play with like this one. Meanwhile, you now have the powers of 2 at hand.

18. Fill in the table below.

2^0 =		2^1 =		2^2 =		2^3 =		2^4 =	
2^5 =		2^6 =		2^7 =		2^8 =		2^9 =	
2^{10} =									

Knowing the use of powers and exponents enables us to establish the rules for the order of operations. This is very important. The expression 8 x 2 + 4 might mean 20 or 48, depending on the order in which you do the arithmetic. For example, if you start with 8 x 2, you get 16. Add that to 4 and you get 20. However, if you add 2 + 4 first, you get 6. 8 x 6 is 48. Clearly, we need to have some rules about what to do first so that we get the same answer every time.

Fortunately, just such a set of rules has been created and agreed to. They are called the "Order of Operations" and are often referred to as PEMDAS. This is just an abbreviation for the words Parentheses, Exponents, Multiplication/Division, Addition/Subtraction. (Some people find it easier to remember as "Please Excuse My Dear Aunt Sally".)

The Order of Operations says that we have to do calculations in a certain order, every time:

1. ***P****arentheses.* Make any calculations inside parentheses before doing anything else.
   Example: 8 x (2 + 4)….Start inside the parentheses with 2 + 4 to get 6. Then multiply by 8 to get 48.

2. ***E****xponents.* Once the parentheses are done (of if there aren't any) calculate any exponents.
   Example: 8 x 3^2 …Start with 3^2, which is 3 x 3, or 9. Then, multiply by eight to get 72.

3. ***M****ultiplication /* ***D****ivision*. Once the exponents are done (or of there aren't any) do any multiplication or division, working from left to right.
   Example: 8 ÷ 4 x 3 …Since we have both multiplication and division, we have to work from left to right. Starting on the left, then, 8 ÷ 4 = 2. Continuing to the right, multiply by 3 to get 6.

4. ***A****ddition /* ***S****ubtraction*. Once the multiplying and dividing are done (or if there isn't any) do any addition or subtraction, working left to right.
   Example: 9 - 5 + 2 …Since we have both addition and subtraction, we have to work from left to right. Starting on the left, then, 9 - 5 = 4. Continuing to the right, add 2 to get 6.

Wow…that's a lot to take in! Let's do an example together before you try some on your own. **You fill in the blanks**.

$$6 \times (12 - 9)^2 - 8 \div 4 + 5 = ?$$

Step 1. ***P****arentheses first*. (12 – 9) = ______ . Now, we have 6 x 3^2 - 8 ÷ 4 + 5 = ?

Step 2. ***E****xponents second*. 3^2 = _______ . Now, we have 6 x 9 - 8 ÷ 4 + 5 = ?

Step 3. ***M****ultiplication and* ***D****ivision third*. Reading our equation left to right, multiplication comes first, and then division. 6 x 9 = ____ and 8 ÷ 4 = ____. Now, we are left with 54 – 2 + 5 = ?

Step 4. ***A****ddition and* ***S****ubtraction last*. Reading our equation left to right, subtraction comes first, and then addition. 54 – 2 = _____. Now, we are left with 52 + 5. So, the final answer is ______ .

19. 7 + 6 x 2 =

20. 47 + 5 x 0 =

21. 4 x (10 ÷ 5) =

22. 8 + (32 – 24) ÷ 4 =

23. 8 x (9 – 7) ÷ 4 + 2 =

24. (8 – 3)^2 x (7 – 3) =

25. 5 + 3^2 x (9 – 4) =

26. 8 + 43 – 8 x 2 – 42 – 6 =

Try a few more with exponents. Remember "P-E-MD-AS". Do the work inside the parentheses first, and then calculate the exponents.

27. $3 \times 5^2 =$

28. $(3 \times 5)^2 =$

29. $3^2 \times 5 =$

30. $3^2 + 5^2 =$

31. $(3 + 5)^2 =$

32. $(3^2 + 5) =$

33. $(3^2 + 5)^2 =$

34. $(3 \times 5)^3 =$

Notice how adding parentheses changes the answer. For example, the equations in questions 27 and 28 use the same base numbers and exponents (3 and 5^2), but the answers are very different because parentheses were added. The same principle can be seen at work in questions 30 and 33.

35. For each equation below, add in parentheses to make the statement true.

a. $24 \div 6 + 6 \times 3 - 3 = 27$

b. $24 \div 6 + 6 \times 3 - 3 = 4$

c. $24 \div 6 + 6 \times 3 - 3 = 0$

d. $24 \div 6 + 6 \times 3 - 3 = 19$

e. $24 \div 6 + 6 \times 3 - 3 = 3$

f. $72 \times 2 \div 6 - 3 \times 2 + 8 \div 4 = 20$

g. $72 \times 2 \div 6 - 3 \times 2 + 8 \div 4 = 16$

h. $72 \times 2 \div 6 - 3 \times 2 + 8 \div 4 = 26$

i. $6 \div 2 \times 3 + 7 \times 8 - 2 = 55$

j. $6 \div 2 \times 3 + 7 \times 8 - 2 = 180$

That was challenging! The next few will be a little easier. Figure out the *numerators* (the part above the dividing line), then the denominators (the part below the line), and then do the division. For example, given $\frac{8 \times 9}{3 \times 4}$

Start with the top part (the numerator): 8 x 9 = 72…

Then do the bottom part (the denominator): 3 x 4 = 12…

Then divide. $\frac{72}{12}$ is the same as 72 ÷ 12, which is equal to 6

Remember to follow P-E-MD-AS very carefully! You should get "easy" answers (no decimals or fractions).

36. $\frac{(36 \div 18 + 6 \times 2)}{7 \times (6 - 4)} =$

37. $(2^2 + 3^2) \div 13 =$

38. $(7 - 5)^2 - (4 - 3)^8 =$

39. $\frac{(17 \times 6 - 4 \times 20) \times 10 \div 5}{8 \times 5 + 4} =$

40. $(5 - 3)^4 \div 4 =$

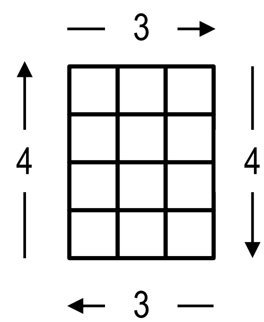
41. $\frac{9 \times 9 + 9}{6 + 36 \div 6 + 3 \times 6} =$

42. $9 \times (3 + 4) \div [(7 + 3) - 3] =$

43. $\frac{2^2 \times (5 + 6)}{25 \div 5 + 3 \times 2} =$

These pages give some practice in the Order of Operations. It is an area which gives many people trouble and which requires a lot of practice. Keep at it! Facility with the order of operations makes all work in mathematics easier.

To conclude this Unit, consider a bit of geometry. Unit 3 concluded with an irregular rectilinear shape, and you found its area. ("Rectilinear" is just a fancy math term that describes a shape made of lines that meet at right angles.) The *perimeter* was never mentioned. Perimeter merely means the distance *around* the figure. For a simple rectangle like the one below, which is 3 units wide and 4 units long, the perimeter would be 3 (across the top) + 4 (down one side) + 3 (across the bottom) + 4 (up the other side) = 14 units total.



Using "P" for perimeter, "W" for width, and "L" for length, this could be stated as P = W + L + W + L. To make it shorter and easier to write, we could also say P = 2W + 2L (two times the width, plus 2 times the length). We could make it even shorter by using parentheses, like this: P = 2(W + L).

P = 2(W + L) is the *formula* for perimeter. We can use it to figure out the perimeter of any rectangle, no matter how big or small! Just change the "W" to the number of units in the width and change the "L" to the number of units in the length. Using the numbers from our rectangle above, we would go from P = 2(W + L) to P = 2(3 + 4). Thus, P(erimeter) = 2 x 7 = 14 units.

At the end of Unit 3, you learned about area, or the number of units *inside* a rectangle. The formula for area is A = WL, or Area = Width x Length. Applied to our rectangle above, A = WL becomes A = 3 x 4 = 12 square units.

44. Using these formulas, find the perimeter and area of the following rectangles:

a. width = 5, length = 7

b. width = 8, length = 15

c. width = 9, length = 4

d. width = 3, length = 48

45. Consider the last rectangle, "d" above. Is there a rectangle with a different width and length, but the same area? Find one.

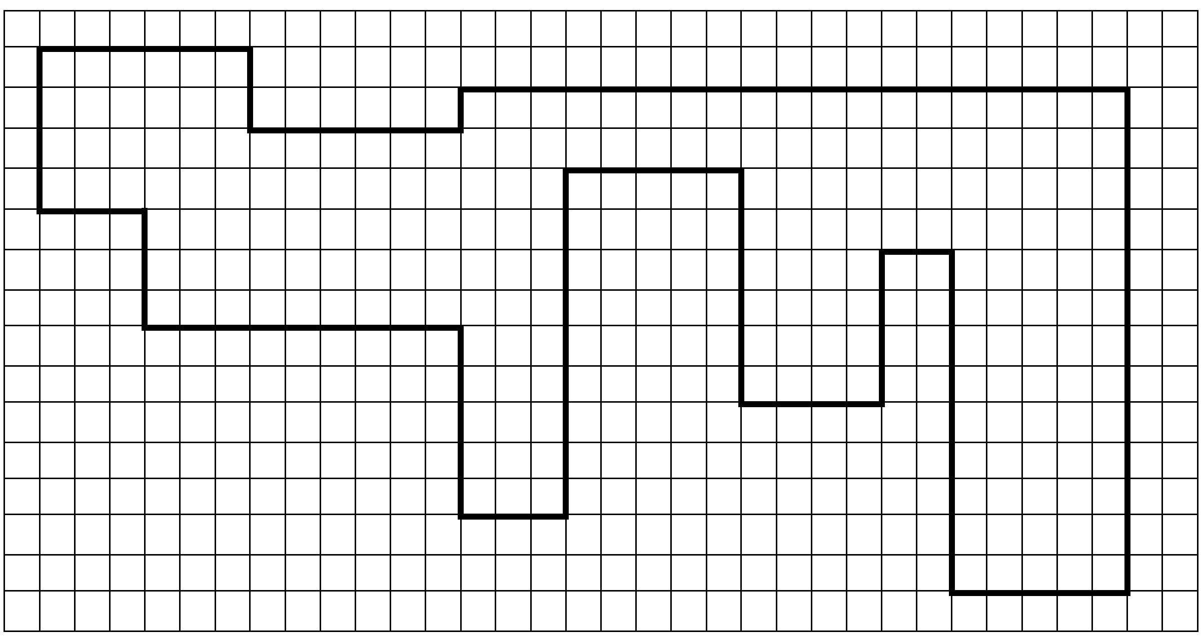
46. In fact, there are eight rectangles with different widths and lengths that have the same area as rectangle "d". Find all eight and give the perimeter for each rectangle. Show your results in the table on the next page.

Width	Length	Perimeter

Remember….all the rectangles must have an _area_ of 144 square units!

47. If you were going to build a rectilinear pen for a herd of sheep, what shape would require the least amount of fence for a given area? Would it be a square, a rectangle, or something like the one below?

The figure from the end of unit 3 appears below. In Unit 3, you found the area, or the number of square units inside the figure.



48. Find the perimeter of this figure, and show the calculations you made to figure it out.

49. Draw some straight lines inside the figure above so that it becomes a group of 7-10 boxes. Find the area of each box using the A = WL formula. Add up the areas of all the boxes. Does the number match what you got in Unit 3?

50. How would you feel about building a house of this shape? Explain why you feel that way.

Since this is your final Unit, we'd appreciate any feedback or suggestions you have for improving the Course!

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Remember: First names only & please let us know if your address changes